THE TIME TRANSFER AND SYNGE WORLD FUNCTIONS IN RPS

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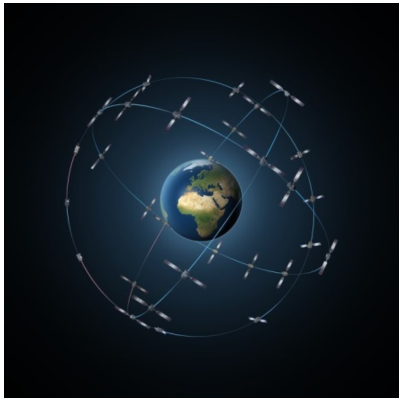
Outline

- Motivation
- Timing corrections
- Time transfer function
- Synge world function
- Simulating null coordinates
- Solving inverse problem
- Still to come

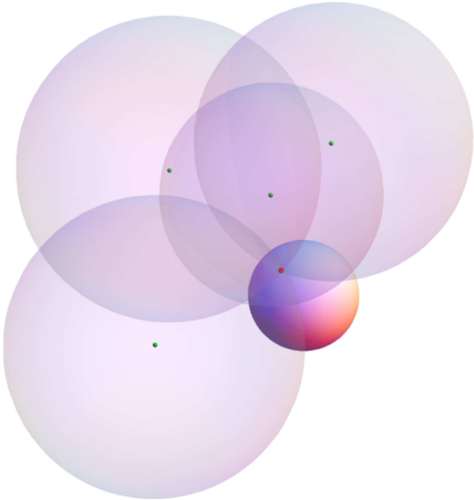
Motivation

GNSS – Global navigation satellite system

GPS,
GLONASS,
GALILEO,
BEIDOU



$$c^2(t_P - t_i) = (x_i - x_P)^2 + (y_i - y_P)^2 + (z_i - z_P)^2$$

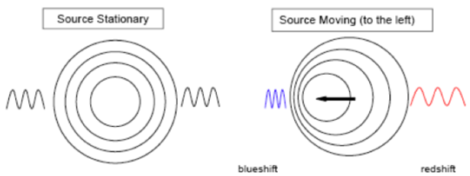


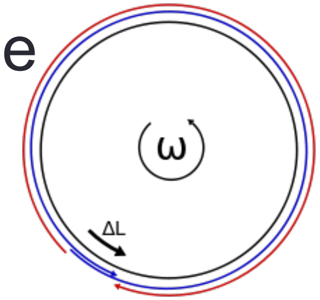
- RPS – a fully relativistic positioning system
- High accuracy with no corrections
- Autonomy – no ground tracking required
- Enables measurements of gravitational field

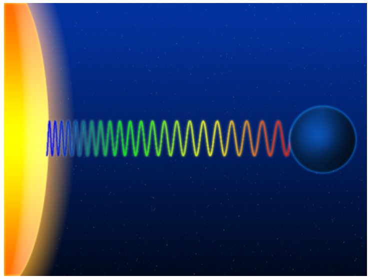
Timing corrections

- Proper time(τ): time indicated by an on-board clock
- Coordinate time (t): time indicated by a clock very far away from any mass

- Requirement: to determine the time of flight of light rays through curved spacetime







- Time transfer and Synge world function relationship

$$\sigma(t, x^i, t', x^{i'}) = \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda 0}^{\lambda_1} g_{ab}(x) \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} d\lambda$$

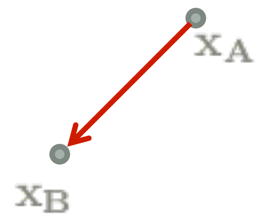
$$\sigma(t, x^i, t - \mathcal{T}(x^i, x^{i'}), x^{i'}) = 0$$

Time transfer function

- Time transfer function

$$\frac{1}{c}\int_{x_B}^{x_A} dx^0 = t_B - t_A = \mathcal{T}_e(t_A, \mathbf{x}_A, \mathbf{x}_B) = \mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B)$$

$$\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B) = \frac{|\mathbf{x}_A - \mathbf{x}_B|}{c} + \sum_{n=1}^{\infty} \mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B)$$



- Post Minkowskian metric

$$ds^2 = \mathcal{A}(r)\left(dx^0\right)^2 - \mathcal{B}^{-1}(dr^2 + r^2 d\theta^2 + r^2 \cos\theta^2 d\phi^2)$$

$$\mathcal{A}(r) = 1 - 2\frac{m}{r} + 2\beta\frac{m^2}{r^2} - \frac{3}{2}\beta_3\frac{m^3}{r^3} + \beta^4\frac{m^4}{r^4} + \sum_{n=5}^{\infty}\frac{(-1)^n n}{2^{n-2}}\beta^n\frac{m^n}{r^n}$$

$$\mathcal{B}^{-1}(r) = 1 + 2\gamma\frac{m}{r} + \frac{3}{2}\epsilon\frac{m^2}{r^2} + \frac{1}{2}\gamma_3\frac{m^3}{r^3} + \frac{1}{16}\gamma_4\frac{m^4}{r^4} + \sum_{n=5}^{\infty}(\gamma_n - 1)\frac{m^n}{r^n}$$

$$\mathcal{U}(r) = \frac{1}{\mathcal{A}(r)\mathcal{B}(r)} = 1 + 2(1+\gamma)\frac{m}{r} + \sum_{n=2}^{\infty} 2\kappa_n\frac{m^n}{r^n}$$

- Quasi Minkowskian light rays

$$\begin{cases} x^0 = ct_a + \xi|\mathbf{x}_B - \mathbf{x}_A| + \sum_{n=1}^{\infty} X^0_{(n)}(\mathbf{x}_A, \mathbf{x}_B, \xi) \\ \mathbf{x} = \mathbf{z}(\xi) + \sum_{n=1}^{\infty} \mathbf{X}_{(n)}(\mathbf{x}_A, \mathbf{x}_B, \xi) \end{cases}$$

Time transfer function

- Solving for dt

$$cdt = \frac{J}{E}d\phi \pm \frac{1}{r}\sqrt{r^2\mathcal{U}(r) - \left(\frac{J}{E}\right)^2}dr$$

- Time transfer function

$$c\mathcal{T}(r_A, r_B, \mu) = \frac{J}{E}(\phi_A - \phi_B) + \int_{r_A}^{r_B} \frac{1}{r}\sqrt{r^2\mathcal{U}(r) - \left(\frac{J}{E}\right)^2}dr$$

Linet & Teyssandier, 2014

- Expanded terms

$$\frac{J}{E} = r_c\left[1 + \sum_{n=1}^{\infty}\left(\frac{m}{r_c}\right)q_n\right] \qquad r_c = \frac{r_A r_B}{R_{AB}}\sin(\phi_A - \phi_B)$$

$$q_n = -c\left(\frac{r_c}{m}\right)^n \frac{\sin(\phi_A - \phi_B)}{r_c}\frac{\partial\mathcal{T}^{(n)}(r_A, r_B, \mu)}{\partial(\cos\phi_A - \phi_B)}$$

$$R_{AB} = \sqrt{r_A^2 + r_B^2 - 2r_A r_B\cos(\phi_A - \phi_B)}$$

Time transfer function

- The time transfer function

$$\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B) = \frac{|\mathbf{x}_A - \mathbf{x}_B|}{c} + \sum_{n=1}^{\infty} \mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B)$$

$$\mathcal{T}^{(1)}(r_A, r_B, (\phi_A - \phi_B)) = m\frac{1+\gamma}{c} \ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right)$$ → Shapiro, 1964

$$\mathcal{T}^{(2)}(r_A, r_B, (\phi_A - \phi_B)) = \frac{m^2}{c}\frac{R_{AB}}{r_A r_B}\left[\kappa_2 \frac{(\phi_A - \phi_B)}{\sin(\phi_A - \phi_B)} - \frac{(1+\gamma)^2}{1+\cos(\phi_A - \phi_B)}\right],$$ → Linet & Teyssandier, 2014

$$\mathcal{T}^{(3)}(r_A, r_B, (\phi_A - \phi_B)) = \frac{m^3}{c}\frac{R_{AB}(r_A + r_B)}{r_A^2 r_B^2 (1+\cos(\phi_A - \phi_B))}\left[\kappa_3 - (1+\gamma)\kappa_2 \frac{(\phi_A - \phi_B)}{\sin(\phi_A - \phi_B)} + \frac{(1+\gamma)^3}{1+\cos(\phi_A - \phi_B)}\right]$$

$$\begin{aligned}\mathcal{T}^{(4)}(r_A, r_B, \mu) = &\frac{m^4}{c}\frac{R_{AB}}{r_A^3 r_B^3 (1-\mu^2)^2}\left[-(1+\gamma)^4 \frac{5\left(2r_A^2 + 2r_B^2 + r_A r_B(3-\mu)\right)(1-\mu)^2}{6(1+\mu)}\right.\\ &+\kappa_2(1+\gamma)^2 \frac{\left(2r_A^2 + 2r_B^2 + r_A r_B(3-\mu)\right)(1-\mu)^2 \arccos\mu}{\sqrt{1-\mu^2}}\\ &-\kappa_2^2 \frac{\arccos\mu\left(R_{AB}^2(1-\mu^2) - (r_B - r_A\mu)(r_B\mu - r_A)\sqrt{1-\mu^2}\arccos\mu\right)}{2\sqrt{1-\mu^2}}\\ &+(1+\gamma)\kappa_3\left((1-\mu)\left(\left(r_A^2 + r_B^2\right)(3-\mu) + 2r_A r_B(1-3\mu)\right) + R_{AB}^2\sqrt{1-\mu^2}\arccos\mu\right)\\ &\left.+\frac{\kappa_4}{2}\left(\left(2r_A r_B - \left(r_A^2 + r_B^2\right)\mu\right)(1-\mu^2) + R_{AB}^2\sqrt{1-\mu^2}\arccos\mu\right)\right]\end{aligned}$$

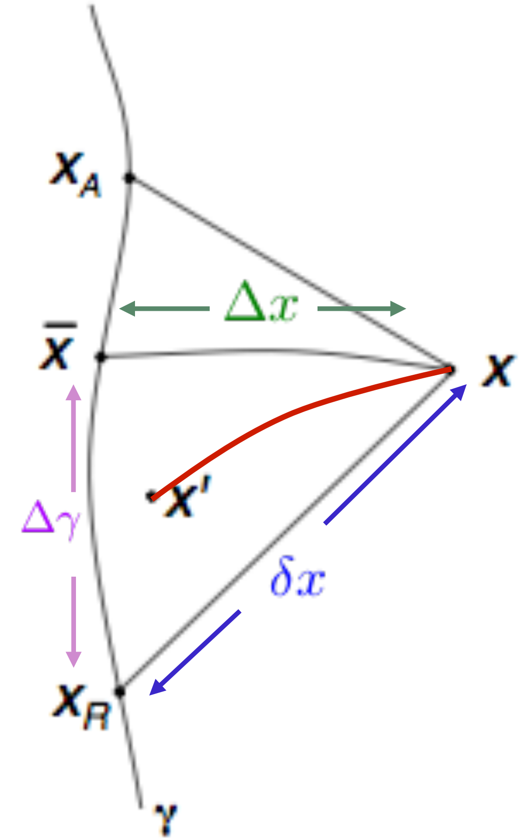
Synge world function

$$\sigma(x,x') = \frac{1}{2}(\lambda_1 - \lambda_0)\int_{\lambda 0}^{\lambda_1} g_{ab}(z)\frac{dz^a}{d\lambda}\frac{dz^b}{d\lambda}d\lambda$$

$$= \frac{1}{2}\xi(\lambda_1 - \lambda_0)^2$$

$\Rightarrow$ **for light** $\sigma = 0$

$$g^{ab}\sigma_a\sigma_b = 2\sigma \qquad g^{a'b'}\sigma'_a\sigma'_b = 2\sigma$$

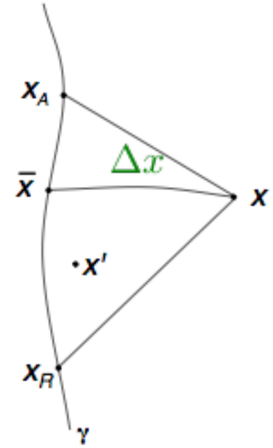


$$\sigma = g_{a'b'}\delta x^{a'}\delta x^{b'}\epsilon^2 + A_{a'b'c'}\delta x^{a'}\delta x^{b'}\delta x^{c'}\epsilon^3 + B_{a'b'c'd'}\delta x^{a'}\delta x^{b'}\delta x^{c'}\delta x^{d'}\epsilon^4 + O(\epsilon^5)$$

$$\gamma(\tau') = \gamma(\bar{\tau}) + u(\bar{\tau})(\tau' - \bar{\tau})\epsilon + \dot{u}(\bar{\tau})\frac{\Delta\tau^2}{2!}\epsilon^2 + O(\epsilon^3)$$

$$\Delta\tau = (\tau' - \bar{\tau}) = \tau_1 + \tau_2\epsilon + \tau_3\epsilon^2 + O(\epsilon^3)$$

Synge world function



- Proper time

$$\tau' = \bar{\tau} + \tau_1 + \tau_2 + \ldots$$

$$\tau_1 = \frac{\bar{r}\Delta \mathrm{r} u^r}{2m - \bar{r}} + \Delta \mathrm{t} \mathrm{E} - \Delta\phi L + \rho$$

$$\tau_2 = -\frac{\Delta \mathrm{r}^3 m\left(\mathrm{E}^2\bar{r}^3 - L^2\bar{r} + 2L^2 m\right)}{2\rho\bar{r}^2\left(\bar{r} - 2m\right)^3} + \Delta \mathrm{r}\left(\frac{\Delta\theta^2\left(\mathrm{E}^2\bar{r}^3 - L^2\bar{r} + 4m\bar{r}^2 - 2\bar{r}^3 + 2L^2 m\right)}{4m\rho\bar{r} - 2\rho\bar{r}^2}\right.$$

$$\left. + \frac{\Delta\phi^2\left(\mathrm{E}^2\bar{r}^3 - 3L^2\bar{r} + 4m\bar{r}^2 - 2\bar{r}^3 + 6L^2 m\right)}{4m\rho\bar{r} - 2\rho\bar{r}^2} - \frac{\Delta\phi L}{\bar{r}}\right)$$

$$+ \Delta \mathrm{t}^2\left(\frac{\Delta \mathrm{r} m\left(\left(2 - 3\mathrm{E}^2\right)\bar{r}^3 + L^2\bar{r} - 4m\bar{r}^2 - 2L^2 m\right)}{2\rho\bar{r}^4\left(2m - \bar{r}\right)} + \frac{\Delta\phi L m u^r}{2\rho\bar{r}^2} - \frac{m u^r}{2\bar{r}^2}\right)$$

$$+ \Delta \mathrm{t}\left(\Delta \mathrm{r}\left(\frac{\Delta\phi \mathrm{E} L\left(\bar{r} - m\right)}{\rho\bar{r}\left(2m - \bar{r}\right)} + \frac{\mathrm{E}m}{\bar{r}\left(\bar{r} - 2m\right)}\right) - \frac{\Delta \mathrm{r}^2 \mathrm{E} m u^r}{2\rho\left(\bar{r} - 2m\right)^2} + \frac{\Delta\theta^2 \mathrm{E}\bar{r} u^r}{2\rho} + \frac{\Delta\phi^2 \mathrm{E}\bar{r} u^r}{2\rho}\right) - \frac{\Delta \mathrm{t}^3 \mathrm{E} m u^r}{2\rho\bar{r}^2}$$

$$+ \Delta \mathrm{r}^2\left(\frac{\Delta\phi L u^r\left(2\bar{r} - 5m\right)}{2\rho\left(\bar{r} - 2m\right)^2} + \frac{m u^r}{2\left(\bar{r} - 2m\right)^2}\right) + \Delta\theta^2\left(\frac{\bar{r} u^r}{2} - \frac{\Delta\phi L\bar{r} u^r}{2\rho}\right) - \frac{\Delta\phi^3 L\bar{r} u^r}{2\rho} + \frac{1}{2}\Delta\phi^2\bar{r} u^r$$

$$\rho^2 = \left(g_{ab}\Delta x^a u^b\right)^2 + g_{ab}\Delta x^a\Delta x^b$$

Simulating null coordinates

- Post Minkowskian to Schwarzschild

$$t' = t,$$
$$r' = r\left[1 - \frac{r_S}{2r} - \sum_{n=2}^{\infty}\left(\frac{r_S}{2^n r}\right)^n\right],$$
$$\theta' = \theta,$$
$$\phi' = \phi.$$

- Schwarzschild circular orbit

$$t(\tau) = \frac{c}{\sqrt{1 - \frac{3r_S}{r_0}}}\tau + t_0,$$
$$r(\tau) = r_0,$$
$$\theta(\tau) = \frac{\pi}{2},$$
$$\phi(\tau) = \sqrt{\frac{c^2 r_S}{2r_0^3\left(1 - \frac{3r_S}{2r_0}\right)}}\tau + \phi_0$$

Delva & Olympio 2011

- Baby test

$$r_0 = 4.2 \times 10^7 \text{ m}, \quad r_P = 5.0 \times 10^7 \text{ m},$$
$$\phi_0 = 0, \quad \theta_P = \pi/2,$$
$$t_0 = 0. \quad \phi_P = 0.$$

- Results

Order	$t_p = 1s \Rightarrow \tau$ [s]
Delva	0.9733148699
TT(1)	0.973314869993433021...
TT(3)	0.973314869993433126 9...
TT(4)	0.973314869993433126 9...
SW(2)	0.973314869993492824 2...
SW(5)	0.973314869993433121 4...
SW(6)	0.973314869993433127 9...

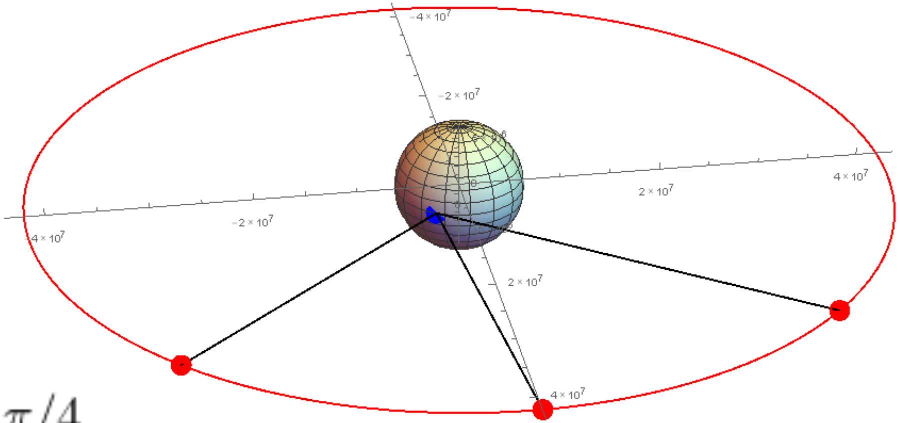
Order	$t_P = 1000s \Rightarrow \tau$ [s]
Delva	999.9710561425
TT(3)	999.97105515857394...
TT(4)	999.971056183339852...
SW(5)	999.971056071208982...
SW(6)	999.971056071208977...

Simulating null coordinates

- Three satellites

$$x_P = (1, 6.3 \times 10^6, \pi/2, -\pi/6)$$

$$t_{0,i} = 0 \quad \forall i,$$
$$r_{0,i} = 4.2 \times 10^7 \quad \forall i,$$
$$\phi_{0,1} = 0, \quad \phi_{0,2} = \pi/4, \quad \phi_{0,3} = -\pi/4$$



- Results

Order	Satellite 1: $t_P = 1s \Rightarrow \tau$ [s]
TT(4)	0.87764941761 61300...
SW(12)	0.87764941761 42050...
SW(11)	0.87764941759 63577...

Order	Satellite 2: $t_P = 1s \Rightarrow \tau$ [s]
TT(4)	0.863819405 26182...
SW(12)	0.863819405 72704...
SW(11)	0.863819406 22337...

Order	Satellite 3: $t_P = 1s \Rightarrow \tau$ [s]
TT(4)	0.88007857144 57471...
SW(12)	0.88007857144 71294...
SW(11)	0.88007857144 74831...

Solving inverse problem

- Expand everything!

$$t_{P,exp} = t_P^{(0)} + \sqrt{\epsilon} t_P^{(1/2)} + \epsilon t_P^{(1)} + \ldots + \epsilon^{order} t_P^{(order)},$$
$$r_{P,exp} = r_P^{(0)} + \sqrt{\epsilon} r_P^{(1/2)} + \epsilon \left(r_P^{(1)} - m \right) + \mathcal{O}(\epsilon^{3/2})$$
$$\phi_{P,exp} = \phi_P^{(0)} + \sqrt{\epsilon} \phi_P^{(1/2)} + \epsilon \phi_P^{(1)} + \ldots + \epsilon^{order} \phi_P^{(order)}.$$

$$t_{i,exp} = \tau + \frac{3m}{2r_0} \tau \epsilon + \mathcal{O}(\epsilon^2),$$
$$r_{i,exp} = r_0 - m\epsilon + \mathcal{O}(\epsilon^2),$$

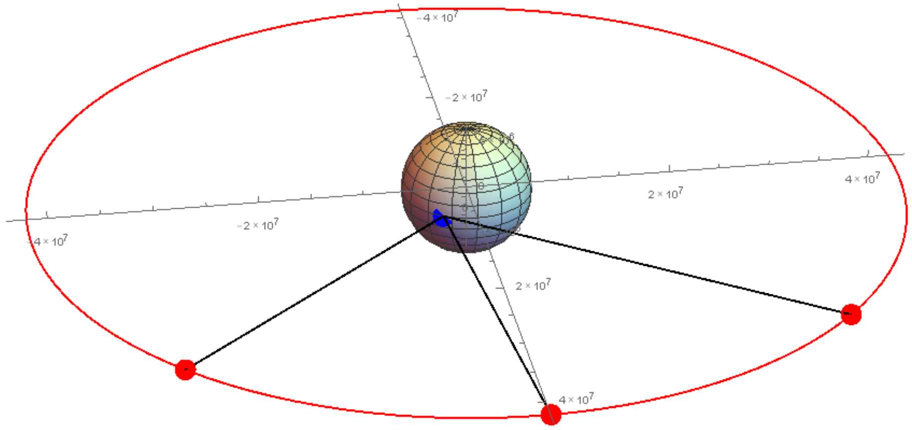
$$\phi_{i,exp} = \phi_{i,0} + \sqrt{\frac{m}{r_0^3}} \tau \sqrt{\epsilon} + \sqrt{\frac{m^3}{4r_0^5}} 3\tau \epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2})$$

$$\begin{aligned}
\mu_{i,exp} =& \cos\left(\phi_i - \phi_P^{(0)}\right) + \sin\left(\phi_{i,0} - \phi_P^{(0)}\right) \left(\phi_P^{(1/2)} - \sqrt{\frac{m}{r_0^3}} c\tau_i\right) \sqrt{\epsilon} \\
& - \left[\cos\left(\phi_{i,0} - \phi_P^{(0)}\right) \frac{\left(\sqrt{\frac{4m}{r_0}} c\tau_i - 2r_0 \phi_P^{(1/2)}\right)^2}{8r_0^2} - \sin\left(\phi_{i,0} - \phi_P^{(0)}\right) \phi_P^{(1)} \right] \epsilon \\
& + \left[\frac{1}{6} \sin\left(\phi_{i,0} - \phi_P^{(0)}\right) \left(\sqrt{\frac{m}{r_0^3}} c\tau_i - \phi_P^{(1/2)}\right)^3 + \cos\left(\phi_{i,0} - \phi_P^{(0)}\right) \left(\sqrt{\frac{m}{r_0^3}} c\tau_i - \phi_P^{(1/2)}\right) \phi_P^{(1)} \right. \\
& \left. + \sin\left(\phi_{i,0} - \phi_P^{(0)}\right) \left(\phi_P^{(3/2)} - \frac{3\left(\frac{2m}{r_0}\right)^{3/2} c\tau_i}{4\sqrt{2} r_0}\right) \right] \epsilon^{3/2} + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Solving the inverse problem

- 3 Satellites

Order	t_P [s]	Error
Actual	1.0	
0	1.000000031160543317958931839335	3.1161×10^{-8}
0.5	0.999999999801722208034042239405	1.9828×10^{-10}
1	1.000000000000000066805491440617	6.6805×10^{-17}
1.5	1.000000000000000000228863072788	2.2886×10^{-19}
2	0.999999999999999999999999718036	2.8196×10^{-25}
2.5	1.000000000000000000000000000133	1.3263×10^{-28}
3	1.000000000000000000000000000000	9.4083×10^{-34}

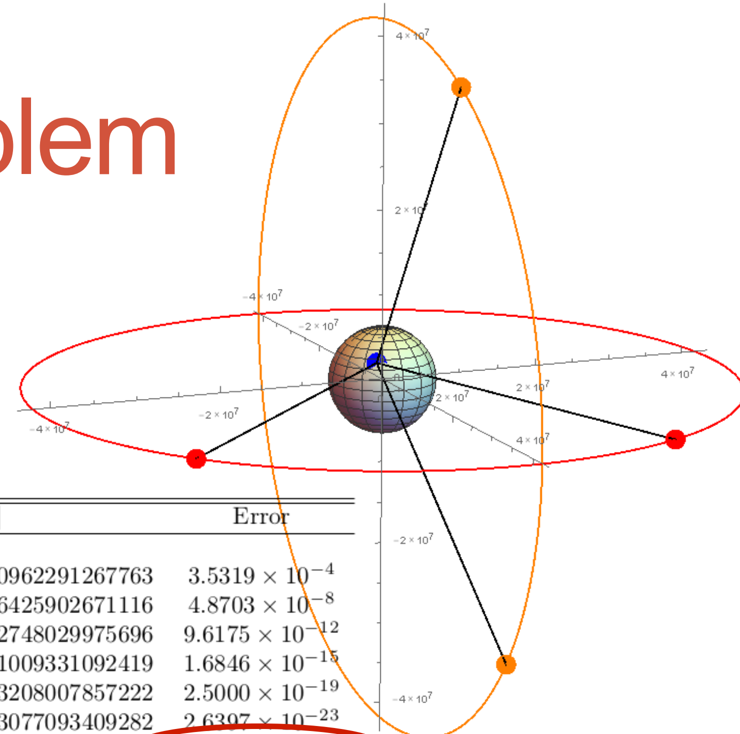


Order	r_P [10^6 m]	Error
Actual	6.3	
0	6.299989882179941576678355635747	-1.6060×10^{-6}
0.5	6.300000000935775873822418454687	1.4854×10^{-10}
1	6.299999999999993758182253996945	9.9076×10^{-16}
1.5	6.299999999999999430350474651 08	9.0421×10^{-18}
2	6.300000000000000000000025068941	3.9792×10^{-24}
2.5	6.299999999999999999999999973536	4.2006×10^{-27}
3	6.300000000000000000000000000000	4.7364×10^{-32}

Order	ϕ_P [rad]	Error
Actual	$-\pi/6$	
0	-0.523663456801297482319083172952	1.1235×10^{-4}
0.5	-0.523598775560885789129469487053	-7.1454×10^{-11}
1	-0.523598775598309328734082371606	1.9969×10^{-14}
1.5	-0.523598775598298889248115917379	3.0884×10^{-17}
2	-0.523598775598298873077052496507	1.0453×10^{-22}
2.5	-0.523598775598298873077107223789	1.2906×10^{-26}
3	-0.523598775598298873077107230547	1.6206×10^{-32}

Solving the inverse problem

- Inclined orbits – 4 satellites



Order	θ_P [rad]	Error
Actual	$\pi/3$	
0	1.04714970663220844904783364679 3	04.5688×10^{-5}
0.5	1.04719757090573894413222590005 7	1.8821×10^{-8}
1	1.04719755119416417340784597919 3	2.3239×10^{-12}
1.5	1.04719755119659773253785987591 7	1.3003×10^{-17}
2	1.04719755119659774625600260563 8	9.7201×10^{-20}
2.5	1.04719755119659774615418069078 4	3.2248×10^{-23}
3	1.04719755119659774615421446810 6	6.6967×10^{-27}

Order	ϕ_P [rad]	Error
Actual	$-\pi/6$	
0	-0.523783708168425960962291267763	3.5319×10^{-4}
0.5	-0.523598750097320206425902671116	4.8703×10^{-8}
1	-0.523598775603334562748029975696	9.6175×10^{-12}
1.5	-0.523598775598297991009331092419	1.6846×10^{-15}
2	-0.523598775598298873208007857222	2.5000×10^{-19}
2.5	-0.523598775598298873077093409282	2.6397×10^{-23}
3	-0.5235987755982988730771072304[illegible]2	1.0518×10^{-28}

Order	r_P [10^6 m]	Error
Actual	6.3	
0	6.29799109576144017215054962461 9	3.1887×10^{-4}
0.5	6.29999628817188546413163437225	5.8918×10^{-8}
1	6.29999999995439596592976583797 4	7.2387×10^{-12}
1.5	6.29999999999999901247203496201 99	1.5675×10^{-15}
2	6.29999999999999999856278512241 1	2.2813×10^{-19}
2.5	6.29999999999999999999973442883 1	4.2154×10^{-23}
3	6.29999999999999999999999999416 63	9.2599×10^{-27}

Order	t_P [s]	Error
Actual	1.0	
0	1.00000472652589847041988166451 1	4.7265×10^{-6}
0.5	1.00000000031712849747729920919 0	3.1713×10^{-10}
1	1.00000000000006844821961611832 2	6.8448×10^{-14}
1.5	1.00000000000000000427656715917 0	4.2766×10^{-18}
2	1.00000000000000000000028969009 2	2.8969×10^{-22}
2.5	0.99999999999999999999999989010 0	1.0990×10^{-25}
3	0.99999999999999999999999999996 7	3.3013×10^{-29}

Solving the inverse problem

- Elliptic orbits – equations of motion

$$\tau = \frac{Y}{2\pi}(1-e^2)^{3/2}\sqrt{1-\frac{m}{l}(3+e^2)}\int d\psi \left\{ \left(\frac{1-e^2}{1-e\cos\psi}\right)^{-2} \left(1-2\frac{m}{l}\left(2+\frac{1-e^2}{1-e\cos\psi}\right)\right)^{-1/2} \frac{\sqrt{1-e^2}}{1-e\cos\psi} \right\}$$

$$r(\psi) = \frac{l}{1+e\frac{e-\cos\psi}{e\cos\psi-1}}$$

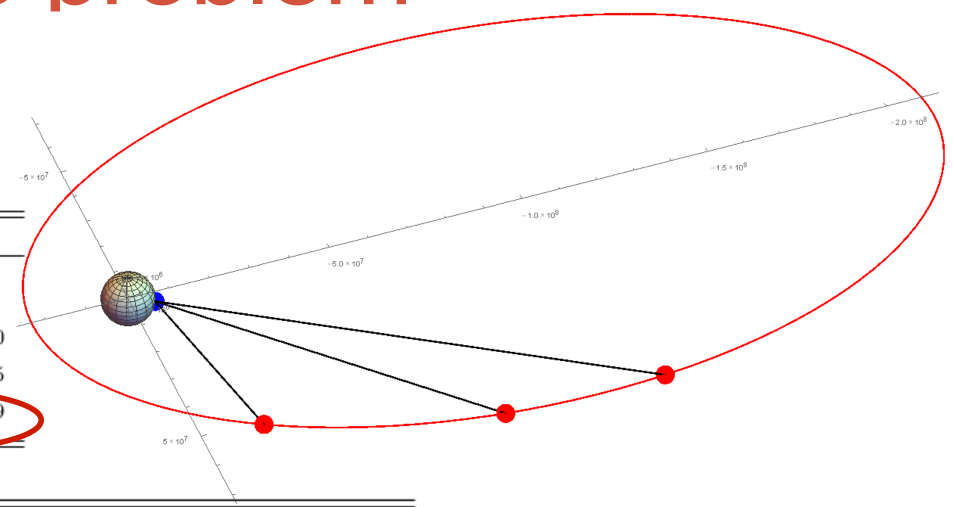
$$t = \frac{Y}{2\pi}(1-e^2)^{3/2}\sqrt{[1-2\frac{m}{l}(1+e)][1-2\frac{m}{l}(1-e)]}$$
$$\int d\psi \left\{ \left(\frac{1-e^2}{1-e\cos\psi}\right)^{-2} \left(1-2\frac{m}{l}\left(\frac{1-e^2}{1-e\cos\psi}\right)\right)^{-1} \left(1-2\frac{m}{l}\left(2+\frac{1-e^2}{1-e\cos\psi}\right)\right)^{-1/2} \frac{\sqrt{1-e^2}}{1-e\cos\psi} \right\}$$

$$Y = 2\pi l\sqrt{\frac{l}{2m}}(1-e^2)^{-3/2}$$

$$\phi = \int d\psi \left\{ \left[1-2\frac{m}{l}\left(3+\frac{e(e-\cos\psi)}{e\cos\psi-1}\right)\right]^{-1/2} \frac{\sqrt{1-e^2}}{1-e\cos\psi} \right\}$$

Solving the inverse problem

- Elliptic orbits – 3 satellites



Order	t_P [s]	Error
Actual	1.0	
0	0.999981147292624478443271196316	1.8852×10^{-4}
1	1.00000000311447240513143137083	3.1144×10^{-10}
2	0.999999999999991902940522415661	8.0971×10^{-15}
3	1.000000000000000000240528310966	2.4052×10^{-19}

Order	r_P [10^6 m]	Error
Actual	6.3	
0	6.301820859016794837474852781689	2.8902×10^{-3}
1	6.299998869872663870009048889219	1.7938×10^{-7}
2	6.29999999725109007327543474487	4.3633×10^{-11}
3	6.300000000000019882419173483962	3.1559×10^{-15}

Order	ϕ_P [rad]	Error
Actual	2.617993877991494411361372840474	
0	2.617433575705034511049060737423	2.1401×10^{-3}
1	2.617993718180396219341713036414	6.1043×10^{-8}
2	2.617993878016665537549958414726	9.6147×10^{-12}
3	2.617993877991538123347549284001	1.6714×10^{-14}

Still to come

- Time transfer method applied to inclined eccentric orbits in solving the inverse problem
- Use Synge world function to produce null coordinates for eccentric orbits
- Determine whether Synge world function is applicable to the inverse problem
- Develop ABC system with these semi-analytic solutions
- Determine whether applicable to real world systems, e.g. the metric used in Andreja Gomboc's talk
- Compare with Andreja and Uros's numerical method
- Read off gravitational parameters
- Applications in relativistic astrophysics???